VI

THE EFFECT OF FRICTION

44. Terrestrial mechanisms are subject to damping forces of a frictional character some of which can be represented approximately as functions of the velocity. Since such forces change sign with the velocity, it is necessary to assume that the frictional force be an odd function of the velocity when we are dealing with vibratory motion, if we are to avoid the mathematical difficulties caused by the presence of a constant which may change its sign during the motion. One example of the combined effect of friction and resonance will be given.

Consider the equation

$$\frac{d^2y}{dt^2} = -\mu\frac{dy}{dt} - \kappa^2 \sin(y - n't - \epsilon'). \tag{44.1}$$

This may be visualized as the equation of motion of a wheel subjected to a periodic couple and also to a frictional couple proportional to the angular velocity dy/dt.

Suppose that the wheel is making complete revolutions and that its angular velocity is always greater than n'. Put

$$y = n't + \epsilon' + x, \tag{44.2}$$

so that the equation can be written

$$\frac{d^2x}{dt^2} + \kappa^2 \sin x = -\mu\left(n' + \frac{dx}{dt}\right). \tag{44.3}$$

According to the hypothesis just made, dx/dt never vanishes and consequently (44.3) may be regarded as the equation of motion of a pendulum making complete revolutions but subjected to a disturbing force represented by the right-hand member of (44.3).

From section 18, the solution when $\mu=0$ is

$$x=l+\frac{\kappa^2}{n^2}\sin l+\cdots,\quad l=nt+\epsilon,\quad n=p-n', \tag{44.4}$$

where p is the mean angular velocity of the wheel, that is, the mean value of dy/dt.

To obtain the solution of (44.3) when μ is not 0, we adopt the method of changing the variable, replacing x by l, n. According to the results of section 24, the equations for finding l, n are

$$\begin{aligned}\frac{dn}{dt}&=-\frac{\mu}{K}\frac{\partial x}{\partial l}\left(n'+n\frac{\partial x}{\partial l}\right),\\ \frac{dl}{dt}&=n+\frac{\mu}{K}\frac{\partial x}{\partial n}\left(n'+n\frac{\partial x}{\partial l}\right),\end{aligned} \tag{44.5}$$

in which the right-hand member of (44.3) has been replaced by $n'+n\partial x/\partial l$, according to (24.2). The value of K is easily calculated if we follow the method used in section 31, and is found to be $K=1-\kappa^4/n^4+\cdots$.

By hypothesis dx/dt, and therefore $\partial x/\partial l$, is always positive. The expression for dn/dt is therefore always negative and n is a decreasing variable. It follows that the periodic portion of x given in (44.4) has a continually increasing amplitude.

The effect of the friction is thus to decrease the mean angular velocity of the wheel but to increase the amplitude of the oscillation about this mean.

45. This motion continues until n $(=p-n')$ becomes so small that the solution (44.4) is no longer valid. In accordance with our earlier discussion on the motion of a pendulum, the discontinuous case is reached. One of two events may

happen: either x changes sign and n goes on decreasing until we reach the periodic solution

$$y = \text{periodic function of } n't+\epsilon',$$

which can satisfy (44.1). The wheel is then oscillating to and fro under the periodic force and the friction: the energy supplied by the force counterbalances that lost by friction.

Or else x begins to oscillate. In this case we have to change to the variables c, l, used for the motion of an oscillating pendulum. As in section 31 we have for the change

$$x = c \sin l + \cdots, \quad n = \kappa\left(1 - \frac{c^2}{16} + \cdots\right),$$

$$K = \kappa c\left(1 - \frac{c^2}{8} + \cdots\right).$$

The equation for c is therefore

$$\frac{dc}{dt} = -\mu \frac{(n' + cn \cos l + \cdots)(c \cos l + \cdots)}{\kappa c\left(1 - \frac{c^2}{8} + \cdots\right)}.$$

The principal part of the non-periodic term in this expression is $-\frac{1}{2}\mu c$, so that c is a variable which is oscillating but whose mean value is decreasing. It follows that x is an oscillating variable with a decreasing amplitude.

When the amplitude of x becomes small it is convenient to return to (44.3). This equation has a particular solution $x = x_0$ where

$$\sin x_0 = -\frac{\mu n'}{\kappa^2},$$

provided $\mu n' < \kappa^2$.

This solution is evidently the limiting case. The wheel is making complete revolutions with the angular velocity n' in resonance with the disturbing force, the energy supplied by the force being exactly counterbalanced by that lost in friction.

46. In order to see how this limiting case is approached, put $x=x_0+x_1$ in (44.3), neglecting squares of x_1. We obtain

$$\frac{d^2x_1}{dt^2}+\mu\frac{dx}{dt}+\kappa^2\cos x_0\cdot x_1=0. \tag{46.1}$$

The solution of this equation is

$$x_1=Ae^{\lambda_1 t}+Be^{\lambda_2 t}, \tag{46.2}$$

where λ_1, λ_2 are the roots of

$$\lambda^2+\mu\lambda+\kappa^2\cos x_0=0,$$

so that

$$\lambda_1,\quad \lambda_2=-\tfrac{1}{2}\mu\pm(\tfrac{1}{4}\mu^2-\kappa^2\cos x_0)^{\frac{1}{2}}. \tag{46.3}$$

There are three cases:—

(i) $\kappa^2\cos x_0>\frac{1}{4}\mu^2$. The square root in λ_1, λ_2 is imaginary and the solution has the form

$$x_1=Ce^{-\frac{1}{2}\mu t}\sin(qt+C_0),\qquad q^2=\kappa^2\cos x_0-\tfrac{1}{4}\mu^2n'^2.$$

(ii) $\frac{1}{4}\mu^2>\kappa\cos x_0>0$. Both roots are real and negative and the solution has the form (46.2) with $\lambda_1<0$, $\lambda_2<0$.

(iii) $\cos x_0<0$. Both roots are real but one is positive and the other negative so that in (46.2), $\lambda_1>0$, $\lambda_2<0$.

In case (i), x_1 approaches zero by oscillations with a decreasing amplitude. In case (ii) the approach to zero is continuous. Case (iii) appears to refer to the unstable solution, namely, to that value of x_0, which is numerically greater than $\pi/2$ in the solution of $\sin x_0=-\mu n'/\kappa^2$.

If $\mu n'>\kappa^2$, the solution $x=x_0$, where x_0 is a constant, does not exist. There may, however, be solutions with period $2\pi/n'$ which do exist. The interest in these cases is, however, mainly mathematical and they will not be further discussed.

The principal feature of the problem is the fact that the resonance case is necessarily reached so that the oscillations have their maximum amplitude. After this stage is passed the amplitude diminishes and, dependent on the initial conditions, one of the types of steady motion is the final outcome. Theoretically these types only exist as t approaches infinity. It should be pointed out, however, that with oscillations of very small amplitude the so-called "statical" friction, which has been neglected, becomes important and actually brings the system to rest or relative rest in a finite time.

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